6.5 Complex Numbers in Polar Form

The Complex Plane

A complex number $z = a + bi$ is represented as a point (a, b) in a complex plane.
The horizontal axis of the complex plane is called the real axis.
The vertical axis of the complex plane is called the imaginary axis.
The imaginary number i is defined as : $i = \sqrt{-1}$ and $i^2 = -1$

Polar Form of a Complex Number

A complex number $z = a + bi$ is said to be in rectangular form.
The polar form of $z = a + bi$ is written as $z = r(\cos\theta + i\sin\theta)$,
where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \dfrac{b}{a}$.

The value of r is called the modulus of the complex number z.
The angle θ is called the argument of the complex number z with $0 \le \theta < 2\pi$.

Write the complex number in polar form.

1) $z = -2 - 2i$ 1) ______________

$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$

$\tan\theta = \dfrac{b}{a} = \dfrac{-2}{-2} = 1$

θ lies in quadrant III, then $\theta = \dfrac{\pi}{4} + \pi = \dfrac{5\pi}{4}$.

$z = 2\sqrt{2}\left(\cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4}\right)$

2) $z = -2 + 2i\sqrt{3}$ 2) ______________

$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$

$\tan\theta = \dfrac{2\sqrt{3}}{-2} = -\sqrt{3}$

θ lies in quadrant II, then $\theta = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$.

$z = 4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

Write the complex number in rectangular form.

3) $\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ 3) ____________

$\sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}$

Product of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their product is given by:
$z_1z_2 = r_1r_2\left[(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$.

Perform the indicated operation. Write the answer in rectangular form.

4) $5(\cos 65° + i\sin 65°) \cdot 4(\cos 175° + i\sin 175°)$ 4) ____________

$20(\cos 240° + i\sin 240°) = 20\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -10 - 10i\sqrt{3}$

5) $6(\cos 164° + i\sin 164°) \cdot 2(\cos 151° + i\sin 151°)$ 5) ____________

$12(\cos 315° + i\sin 315°) = 12\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 6\sqrt{2} - 6i\sqrt{2}$

Quotient of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their quotient is given by $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$.

Perform the indicated operation. Write the answer in rectangular form.

6) $\frac{8(\cos 185° + i\sin 185°)}{4(\cos 50° + i\sin 50°)}$ 6) ____________

$2(\cos 135° + i\sin 135°) = 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} + i\sqrt{2}$

7) $\dfrac{6\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)}{12\left(\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}\right)}$ 7) ______________

$= \frac{1}{2}\left(\cos\left(\frac{3\pi}{2} - \frac{5\pi}{2}\right) + i \sin\left(\frac{3\pi}{2} - \frac{5\pi}{2}\right)\right)$

$= \frac{1}{2}(\cos(-\pi) + i \sin(-\pi)) = \frac{1}{2}(-1 + 0) = -\frac{1}{2}$

DeMoivre's Theorem

Let $z = r(\cos \theta + i \sin \theta)$ be a complex number in polar form. If n is a positive integer, then $z^n = [r(\cos \theta + i \sin \theta)]^n = r^n(\cos(n\theta) + i \sin(n\theta))$.

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Use De Moivre's theorem to simplify the expression.
Write the answer in rectangular form.

8) $[2(\cos 20^\circ + i \sin 20^\circ)]^6$ 8) ______________

$2^6(\cos(120^\circ) + i \sin(120^\circ)) = 64\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -32 + 32i\sqrt{3}$

9) $\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)\right]^5$ 9) ______________

$\sqrt{2}^5\left(\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4}\right) = 4\sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -4 - 4i$

6.5 Exercises pg 736

(17, 29, 37, 47, 53) (23, 35, 39, 49, 59)

6.6 and 6.7 Vectors in the Plane and Dot Products

Vector

A vector is a directed line segment.

The vector $\overrightarrow{PQ}$ has initial point P and terminal point Q.

Vectors are often denoted by boldface letters, such as v.

Component Form of a Vector

The component form of the vector with initial point P = (p_1, p_2) and terminal point Q = (q_1, q_2) is $\overrightarrow{PQ} = \langle q_1 - p_1,\ q_2 - p_2 \rangle = \langle v_1, v_2 \rangle$ = v.

Magnitude of a Vector

The magnitude (or length) of v is $\|v\| = \sqrt{v_1^2 + v_2^2}$.

Equal Vectors

Two vectors v and w are equal if they have the same magnitude and the same direction.

Find the component form and magnitude of the the vector v .

10) **Initial point P (4, –7) and terminal point Q (–1, 5).** 10) ____________

$\overrightarrow{PQ} = \langle q_1 - p_1,\ q_2 - p_2 \rangle = \langle -1 - 4, 5 + 7 \rangle = \langle -5, 12 \rangle$.

$\|\mathbf{v}\| = \sqrt{25 + 144} = 13$.

Let u = $\langle u_1, u_2 \rangle$ and v = $\langle v_1, v_2 \rangle$ be vectors and let k be a scalar (a real number).

Vector Addition

The sum of u and v is the vector u + v = $\langle u_1 + v_1, u_2 + v_2 \rangle$.

The vector u + v is called the resultant vector. To find such vector:

1. **Position u and v, so that the terminal point of u coincides with the initial point of v.**
2. **The resultant vector, u + v, extends from the initial point of u to the terminal point of v.**

Find the specified vector or scalar.

11) **u = $\langle -2, -6 \rangle$ and v = $\langle -6, 8 \rangle$** 11) ____________

u + v = $\langle u_1 + v_1, u_2 + v_2 \rangle = \langle -2 - 6, -6 + 8 \rangle = \langle -8, 2 \rangle$.

Vector Subtraction

The difference of u and v is the vector $\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$.

Find the specified vector or scalar.

12) $\mathbf{u} = \langle -4, -2 \rangle$ and $\mathbf{v} = \langle 6, 7 \rangle$ 12) ____________

$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2 \rangle = \langle -4 - 6, -2 - 7 \rangle = \langle -10, -9 \rangle$.

Scalar Multiplication

If k is a real number and v a vector, the vector kv is called a scalar multiple of the vector v. Then, $k\mathbf{v} = k\langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle$.

Multiplying a vector by any positive real number changes the magnitude of the vector, but not its direction.

Multiplying a vector by any negative real number changes the magnitude of the vector, and reverses its direction.

Find the specified vector or scalar.

13) $\mathbf{v} = \langle -4, -2 \rangle$. 13) ____________

$2\mathbf{v} = \langle 2v_1, 2v_2 \rangle = \langle -8, -4 \rangle$

Find the specified vector or scalar.

14) $2\mathbf{v} - 3\mathbf{w}$ if $\mathbf{v} = \langle -2, 5 \rangle$ and $\mathbf{w} = \langle 3, 4 \rangle$. 14) ____________

$2v - 3w = 2\langle -2, 5 \rangle - 3\langle 3, 4 \rangle = \langle -4, 10 \rangle - \langle 9, 12 \rangle$

$= \langle -4 - 9, 10 - 12 \rangle = \langle -13, -2 \rangle$.

The i and j Unit Vector

The vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ are called standard unit vectors.

Vector i is the unit vector whose direction is along the positive x–axis.

Vector j is the unit vector whose direction is along the positive y–axis.

Linear combination of the vectors i and j

$\mathbf{v} = \langle v_1, v_2 \rangle = v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$

The vector sum $v_1\mathbf{i} + v_2\mathbf{j}$ is called a linear combination of the vectors i and j.

Write the vector u as a linear combination of unit vectors.

15) **Initial point P(2, –5) and terminal point Q(–1, 3).** 15) ____________

$\langle u_1, u_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle -1 - 2, 3 + 5 \rangle = \langle -3, 8 \rangle$

$u_1 i + u_2 j = -3i + 8j$

Find the specified vector operation.

16) **$2\mathbf{u} - 3\mathbf{v}$ if $\mathbf{u} = -3\mathbf{i} + 8\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$.** 16) ____________

$2\mathbf{u} - 3\mathbf{v} = 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j}) = -6\mathbf{i} + 16\mathbf{j} - 6\mathbf{i} + 3\mathbf{j} = -12\mathbf{i} + 19\mathbf{j}$

Direction Angle

If $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j}$ is a vector such that θ is the angle from the positive x –axis to vector v, then θ is called the direction angle of the vector v.
The vector v can be written as $\mathbf{v} = \|\mathbf{v}\| \langle \cos\theta , \sin\theta \rangle$.

To find θ use the rule: $\tan\theta = \dfrac{v_2}{v_1}$.

Find the direction angle θ .

17) $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ 17) ____________

$$\tan\theta = \frac{v_2}{v_1} = \frac{-4}{3}$$

$$\theta' = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ \quad \rightarrow \quad \theta = 360^\circ - 53.1^\circ = 306.9^\circ$$

Resultant Force

A vector that represents a pull or push of some type is called a force vector.
If F_1 and F_2 are two forces acting on an object, the net effect is the same as the resultant force $F = F_1 + F_2$ acted on the object.

Solve the problem.

18) **Two forces, F_1 and F_2 , of magnitude 10 and 30 pounds, respectively, act on an object. The direction of F_1 is N20°E and the direction of F_2 is N65°E. Find the magnitude and the direction of the resultant force.** 18) ____________

Solution:

$$F_1 = \langle \|F_1\|\cos\theta , \|F_1\|\sin\theta \rangle = \langle 10\cos 70^\circ, 10\sin 70^\circ \rangle = \langle 3.4, 9.4 \rangle.$$

$$F_2 = \langle \|F_2\|\cos\theta , \|F_2\|\sin\theta \rangle = \langle 30\cos 25^\circ, 30\sin 25^\circ \rangle = \langle 27.2, 12.7 \rangle.$$

$$F = F_1 + F_2 = \langle 3.4, 9.4 \rangle + \langle 27.2, 12.7 \rangle = \langle 30.6, 22.1 \rangle.$$

$$\|F\| = \sqrt{(30.6)^2 + (22.1)^2} = \sqrt{1424.77} = 37.7 \text{ pounds.}$$

$$\tan\theta = \frac{22.1}{30.6} = 0.7222 \quad \rightarrow \quad \theta = \tan^{-1}(0.7222) = 35.8^\circ$$

Dot Product

The dot product of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$

Find each dot product.

19) **Let $\mathbf{u} = \langle -6, -5 \rangle$, $\mathbf{v} = \langle 3, -4 \rangle$ and $\mathbf{w} = \langle -4, 6 \rangle$.** 19) ____________

a. $\mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{v}$ **b.** $(2\mathbf{u} \cdot 3\mathbf{v})\mathbf{w}$

Solution:

a. $u \cdot w = (-6)(-4) + (-5)(6) = 24 - 30 = -6$

$v \cdot v = (3)(3) + (-4)(-4) = 9 + 16 = 25$

$u \cdot w - v \cdot v = -6 - 25 = \mathbf{-31}$

b. $2u \cdot 3v = \langle -12, -10 \rangle \cdot \langle 9, -12 \rangle$

$= (-12)(9) + (-10)(-12) = -108 + 120 = 12$

$(2u \cdot 3v)w = 12 \langle -4, 6 \rangle = \langle \mathbf{-48}, \mathbf{72} \rangle$

Angle Between Two Vectors

The angle between two vectors u and v is the angle θ, $0 \le \theta \le \pi$, such that

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

Find the angle between the given vectors to the nearest tenth of a degree.

20) **<4, 2> , <3, 5>** 20) ____________

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

$$= \frac{(4)(3) + (2)(5)}{\sqrt{(4)^2 + (2)^2}\,\sqrt{(3)^2 + (5)^2}} = \frac{22}{\sqrt{20}\,\sqrt{34}} \approx 0.8437$$

$\rightarrow \theta = \cos^{-1}(0.8437) = 32.5°$

Orthogonal and Parallel Vectors

The vectors u and v are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.
The vectors u and v are *parallel* if $\mathbf{u} = k\mathbf{v}$.

Determine whether the vectors are parallel, orthogonal, or neither.

21) $\mathbf{v} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{w} = \mathbf{i} - 3\mathbf{j}$ 21) ____________

$\mathbf{v} \cdot \mathbf{w} = (3)(1) + (1)(-3) = 3 - 3 = 0$

$\rightarrow$ **The vectors are orthogonal.**

22) $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{w} = 4\mathbf{i} - 2\mathbf{j}$ 22) ____________

$\mathbf{w} = 2\mathbf{v}$ $\rightarrow$ **The vectors are parallel.**

Work

The work, W, done by a force F moving an object from A to B is

$W = \|F\| \cdot \|\overrightarrow{AB}\| \cdot \cos\theta$,

where $\|F\|$ is the magnitude of the force, $\|\overrightarrow{AB}\|$ is the distance over which the constant force is applied, and θ is the angle between the force and the direction of motion. Work is often measured in foot-pounds.

Solve the problem.

23) A child pulls a sled along level ground by exerting a force of *30 pounds* on a rope that makes an angle of *35°* with the horizontal. How much work is done pulling the sled *200 ft* ? 23) ____________

$W = \|F\| \cdot \|\overrightarrow{AB}\| \cdot \cos\theta = (30)(200)(\cos 35°) = 4915$ foot-pounds.

6.6 Exercises pg 750 (9, 15, 49, 53, 61, 71) (55, 73)
6.7 Exercises pg 719 (3, 13, 17, 25) (39, 43, 55)